

## Predicate Logic

**Idea:** instead of atomic propositions  $a_i$  more complex internal structure allowed at lowest levels:

$p(x), p'(x, y), p''(x, y, z)$  predicate, "function"

**Predicate:** possesses

- finite number of arguments
- each argument comes from a (not necessarily finite) domain
- a predicate assumes values  $\{f, t\}$

1

## Quantors

**Quantors:** new concept not existing this way in propositional logic; makes general statements about properties of a predicate over a whole domain

**Rough Motivation:** assume predicate  $p(x)$  with finite domain (consisting of atomic constants)  $\{x_1, x_2, \dots, x_n\}$ .

2

## Quantors II

**All Quantor:** Consider then the expression

$$p(x_1) \wedge p(x_2) \wedge \dots \wedge p(x_n) = \bigwedge_x p(x)$$

This expression, interpreted naively as in propositional logic, is true iff  $p(x)$  is true *for all*  $x$  from the domain. In predicate logic, this concept, that is, being able to say that the predicate  $p(x)$  is true *for all*  $x$  in the domain generalises to arbitrary (not just finite) domains and is denoted by

$$\forall_x p(x)$$

**We say:**  $\bigwedge_x p(x) \equiv \forall_x p(x)$ : for all  $x$

3

## Quantors III

**Existence Quantor:** Consider expression

$$p(x_1) \vee p(x_2) \vee \dots \vee p(x_n) = \bigvee_x p(x)$$

This expression, interpreted naively as in propositional logic, is true iff  $p(x)$  is true *for some*  $x$  from the domain. In that case we also say, there exists an  $x$  from the domain for which  $p(x)$  is true. This concept generalises to arbitrary (not just finite) domains and is denoted by

$$\exists_x p(x)$$

**We say:**  $\bigvee_x p(x) \equiv \exists_x p(x)$ : for some  $x$ , there exists an  $x$

4

## Semi-Formal Notation

### Syntax of Predicate Logic Formula:

$\langle \text{term} \rangle ::= c | x$  (atomic constant|variable)  
 $\langle \text{term\_sequence} \rangle ::= \langle \text{term} \rangle$   
 $\langle \text{atomic\_formula} \rangle ::= p(\langle \text{term\_sequence} \rangle)$  predicate  $p$   
 $\langle \text{formula} \rangle ::= \langle \text{atomic\_formula} \rangle$   
 $\quad | \forall x(\langle \text{formula} \rangle)$  ( $x$  atomic variable)  
 $\quad | \exists x(\langle \text{atomic\_formula} \rangle)$   
 $\quad | \neg(\langle \text{atomic\_formula} \rangle)$

5

**Interpretation:** set of elements (domain of interpretation)  $Dom$  together with mapping for each predicate

$$p \mapsto Dom_p \subseteq \underbrace{Dom \times \dots \times Dom}_n \text{ times, if } p \text{ } m\text{-predicate}$$

**Valuation:**  $V$  which maps variables

$$x \mapsto v \in Dom$$

**One has:**

$$T_{IV} [p(x_1, \dots, x_m)] \Leftrightarrow p(V(x_1), \dots, V(x_m))$$

$$T_{IV} (\neg F) \Leftrightarrow \neg T_{IV} (F)$$

etc. for  $\wedge, \vee,$

$$T_{IV} [\forall x F] \Leftrightarrow \forall x \in Dom F \text{ replace non-} x \text{ variables } z$$

**In the following:** consider only fully quantified predicate logic formulas, i.e. variables are quantified *exactly* once.

6

## Truth

- $F$  is true (tautology) if  $T_I V(F) = \tau$  for all interpretations
- falseness and other concepts, like *model*, analogous to predicate logic

7

## Relations

**Note:** if  $F(x_1, \dots, x_n)$  is true (tautology), then  $\forall x_1 \forall x_2 \dots \forall x_n F(x_1, \dots, x_n)$  is a tautology.

**Closed Expressions:** no free variables

$$\neg \forall x F(x) = \exists x \neg F(x)$$

$$F_1 \wedge \forall x F_2(x) = \forall x (F_1 \wedge F_2(x))$$

$$F_1 \vee \forall x F_2(x) = \forall x (F_1 \vee F_2(x))$$

$$\forall x F(x) \vee \exists x \neg$$

**Prenex Form:**

$$\forall x \exists y \forall z \dots F(x, y, z)$$

8

## **Modus Ponens**

**Observation:** many principles of propositional logic carry over to predicate logic.

$$\frac{a \Rightarrow b, a}{b}$$

1. All men must die.
2. Socrates is a man.
3. Therefore, Socrates must die.