Propositional and Predicate Logic

A Sightseeing Overview

Introductory Remarks: present overview uses pragmatic, not formalistic perspective. It will apply a naive approach to logics, putting emphasis on understanding the ideas instead of slavish development of the formalism.

Observation: natural languages have

operators	combining	sentences
operators	evaluating	truth

Example:

$$\underbrace{\underbrace{\text{the sky is blue}}_{\text{true}}\ \text{and}\ \underbrace{\text{the grass is green}}_{\text{true}}}_{\text{conjunction}}.$$

Note: example is true exactly if (if and only if – *iff*) subsentences are both true

We have:

conjunctions	"and"
disjunctions	"or"
negation	"not"

Boolean Propositional Logic

Observation:

- propositions have definite truth value (*true, false*)
- propositions can be combined and modified via propositional operators (*and, or, not*)
- the total truth value of a proposition is determined solely by truth value of its subpropositions
- recursive application of preceding rule allows to determine the total truth value of a proposition by evaluating the truth value for the elementary (atomic) subpropositions
- the exact nature of the atomic subpropositions is irrelevant, only their truth value counts

Some Definitions

Def. (Truth Values): the elements from the set {f(alse), t(rue)} are called *truth values*.

Def. (Atomic Propositional Variable): an atomic subproposition whose truth value has not been established, will be called *atomic propositional variable*.

Def.: let PF be shorthand for a propositional formula and APV shorthand for atomic propositional variable

Syntax of Propositional Formula:

$$\begin{aligned}\langle \mathtt{PF}\rangle ::= & \langle \mathtt{APV}\rangle \\ & \mid \langle \mathtt{PF}\rangle \wedge \langle \mathtt{PF}\rangle && \text{(and)} \\ & \mid \langle \mathtt{PF}\rangle \vee \langle \mathtt{PF}\rangle && \text{(or)} \\ & \mid \neg\langle \mathtt{PF}\rangle && \text{(not)} \\ & \mid \mathsf{t} \\ & \mid \mathsf{f}\end{aligned}$$

Interpretation

Def.: Let $A(F) := \{a_1, \ldots, a_n\}$ be the set of APV in (propositional) formula F.

Then a map

$$T_I : A(F) \to \{\mathsf{f}, \mathsf{t}\}$$

assigning to each $a_i \in A(F)$ a truth value, e.g.

$$\begin{aligned} a_1 &\mapsto \mathsf{t} \\ a_2 &\mapsto \mathsf{f} \\ a_3 &\mapsto \mathsf{t}\end{aligned}$$

is an *interpretation*.

Interpretation 2

Remark: an interpretation T_I induces a truth value for a formula via recursive application of

$$T_I(F_1 \wedge F_2) = T_I(F_1) \wedge T_I(F_2)$$
$$T_I(F_1 \vee F_2) = T_I(F_1) \vee T_I(F_2)$$
$$T_I(\neg F) = \neg T_I(F)$$

where F, F_1, F_2 are PF and truth values are combined according to the Boolean algebra

		$\wedge$	$\vee$
f	f	f	f
f	t	f	t
t	f	f	t
t	t	t	t

	$\neg$
f	t
t	f

Boolean Algebra & Logical Expressions

Commutative Law:

$$p \wedge q = q \wedge p$$
$$p \vee q = q \vee p$$

Associative Law:

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$
$$p \vee (q \vee r) = (p \vee q) \vee r$$

Distributive Law:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

Tautology:

$$p \wedge \mathsf{t} = p$$
$$p \vee \mathsf{f} = p$$

Contradiction:

$$p \wedge \mathsf{f} = \mathsf{f}$$
$$p \vee \mathsf{t} = \mathsf{t}$$

Idempotency:

$$p \wedge p = p$$
$$p \vee p = p$$

De Morgan:

$$\neg(p \wedge q) = \neg p \vee \neg q$$
$$\neg(p \vee q) = \neg p \wedge \neg q$$

Boolean Algebra II

Example:

$$\begin{array}{ccccccc} a_1 & \vee (& a_2 & \wedge & a_3 &) =? \\ \uparrow & & \uparrow & & \uparrow & \\ \mathsf{f} & & \mathsf{t} & & \mathsf{t} & \end{array}$$

Task: recapitulate rules of Boolean algebra

Propositional Proofs

Example: Let

$$a_1 = a_2 \wedge \neg a_3$$

with $a_1, a_2, a_3 \in \{\mathsf{f}, \mathsf{t}\}$ Boolean values. Then

$$\begin{aligned} a_1 \vee a_3 &= (a_2 \wedge \neg a_3) \vee a_3 \\ &\overset{\text{distr. law}}{=} (a_2 \vee a_3) \wedge (\neg a_3 \vee a_3) \\ &= (a_2 \vee a_3) \wedge \mathsf{t} \\ &= a_2 \vee a_3 \end{aligned}$$

As formula:

$$a_1 = a_2 \wedge \neg a_3 \Rightarrow a_1 \vee a_3 = a_2 \vee a_3$$

Def. (Equivalence): two propositional formulas are *equivalent* if their truth values are identical under all interpretations.

Question: what is the equivalent propositional logic expression to "$\Rightarrow$"? It turns out that

$$F_1 \Rightarrow F_2 \equiv \neg F_1 \vee F_2$$

("is *equivalent* to") What is "$=$" in propositional logic? $F_1 = F_2$ is equivalent to

$$(F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2)$$

(disjunctive normal form)

Remark: to be able to formulate the concept of proof, a notion of truth is necessary

Def. (Tautology): A formula F is a *tautology* (true) if $T_I(F) = \mathrm{t}$ for all interpretations T_I.

Def. (Contradiction): A formula F is a *contradiction* (false) if $T_I(F) = \mathrm{f}$ for all interpretations T_I.

Def. (Model): A *model* for a set of formulas $\{F_1, \ldots, F_n\}$ is an interpretation T_I such that $T_I(F_i) = \mathrm{t}$ for all $i = 1 \ldots n$. We say that

$$T_I \models F$$

Task: list all the models for the set of Boolean formulas

$$a_1 \Rightarrow a_2, \quad a_1 = a_2$$

Propositional Proofs II

Def. (Propositional Proof): Given a set of premises $\{F_1, \ldots, F_n\}$ (which are assumed as true), a proof of a formula F is a finite sequence of *deduction steps* leading from the set of premises to F.

A deduction step uses the equations of Boolean algebra to construct new formulas which become true under all models for the premises. These formulas are added to the premise set and further deduction steps are applied iteratively.

Propositional Proofs III

Def. (Provability): We say that F is provable from the premises $\{F_1, \ldots, F_n\}$ if there is a proof of F from the premises, or, stated differently, if $F_1 \wedge \cdots \wedge F_n \Rightarrow F$ can be reduced to t by a finite number of steps applying the rules of Boolean algebra (for a way of constructing such a proof path, see the resolution algorithm in later lectures).

Proofs: as alternatives to achieve a proof, one can use

1. truth tables

2. deduction

Deduction

Deduction:

- given a set of *assumptions*, i.e. set of formulas $\{F_1, \ldots, F_n\}$ together with a model T_I

- create sequence of growing sets $\{F_1, \ldots, F_{n_k}\} \supseteq \cdots \supseteq \{F_1, \ldots, F_n\}, k = 1, 2, \ldots$ for which T_I continues to be a model; i.e. all formulas in these sets are true under the model T_I

Rules of Deduction

$\wedge$ elimination:

$$\frac{a \wedge b}{a}$$

and

$$\frac{a \wedge b}{b}$$

$\wedge$ introduction:

$$\frac{a, b}{a \wedge b}$$

$\vee$ elimination:

$$\frac{a \Rightarrow c, b \Rightarrow c, a \vee b}{c}$$

$\vee$ introduction:

$$\frac{a}{a \vee b}$$

and

$$\frac{b}{a \vee b}$$

Modus ponens:

$$\frac{a, a \Rightarrow b}{b}$$

Double negation:

$$\frac{\neg\neg a}{a}$$

Propositional Proofs IV

Def. (Entailment): We say that $\{F_1, \ldots, F_n\}$ entail F if every model for $\{F_1, \ldots, F_n\}$ is a model for F, written

$$\{F_1, \ldots, F_n\} \models F$$

Remark:

1. The proof system for propositional logic is *consistent*, i.e. if F is provable from $\{F_1, \ldots, F_n\}$, then $\{F_1, \ldots, F_n\}$ entails F.
2. Propositional logic is also *complete*, i.e. if there is a proof for F given $\{F_1, \ldots, F_n\}$, it follows that $\{F_1, \ldots, F_n\}$ entails F.
3. While propositional logic is complete, predicate logic, which will be treated later, is not. In predicate logic, there exist true theorems that are not provable. This is one of the important results of Gödel.

Task: There are deep connections between Gödel's Theorem, the Turing Halting problem and Cantor's diagonal argument. Read about those topics.